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BUCKLING IN AN ELASTIC ROD UNDER A TIME-VARYING LOAD

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A rod subject to a steady heavy load [1] may be replaced by a system with one degree of freedom provided that the motion is examined over a sufficiently long time interval [2]. The rod has to be approximated by a system with a larger number of degrees of freedom [3] if the strong load is aperiodic.

The following equation describes the buckling in a homogeneous elastic rod subject to an alternating heavy load:

$$EIw_{xxxx} + N(t)w_{xx} + \rho F w_{tt} = f(x),\ 0 \leqslant x \leqslant L,\ t > 0, \tag{1}$$

where w is the normal deflection, x and t are the longitudinal coordinate and time, L is rod length, ρ is density, F and I are the constant cross-sectional area and bending rigidity of the rod, E is Young's modulus, N(t) is the given longitudinal force (Fig. 1), and f(x) is a function determined by the given small perturbations or imperfections.

Here N(t) is a continuous monotonically increasing function of time, which increases from zero and runs successively through the critical values for the static case, $N(t) = m^2 N_e$, $N_e = \pi^2 EIL^{-2}$ (m = 1, 2, ...).

We assume that the hinge-supported rod is at rest before loading; then the initial and boundary conditions take the form

$$w = w_t = 0\ (t=0,\ 0 \leqslant x \leqslant L),\ w = w_{xx} = 0\ (x = 0, L,\ t > 0). \tag{2}$$

The solution to (1) and (2) is sought as

$$w = \sum_{m=1}^{\infty} q_m(t) \sin \frac{m\pi x}{L}. \tag{3}$$

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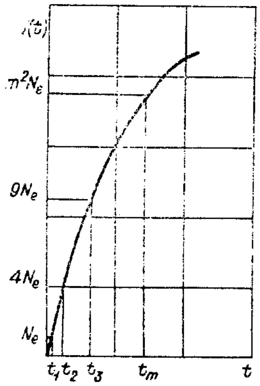

Fig. 1

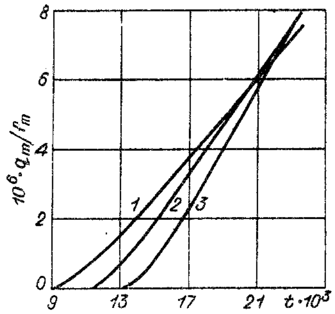

Fig. 2

We substitute (3) into (1) and perform appropriate steps to get $q_m(t)$ in terms of an equation subject to zero initial conditions:

$$q''_m - \lambda^2\alpha_m^2(t)\,q_m = f_m,\ q_m = q'_m = 0,\ t = 0\ \ (m = 1, 2, \ldots),$$
$$\lambda^2 = \frac{\pi EI}{\rho FL},\ \alpha_m^2(t) = -m^2[m^2 - \eta^2(t)],\ \eta^2(t) = \frac{N(t)}{N_e}, \tag{4}$$
$$f_m = \frac{2}{\rho FL}\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx,$$

where $\eta(t)$ characterizes the extent of the loading, N_e is the Euler load, and λ is a large parameter.

As $N(t)$ increases from zero up to m^2N_e, the coefficient α_m^2 for form m remains negative; the instant $t = t_m$, when $N(t) = m^2N_e$ and $\alpha_m^2 = 0$, is a turning point for (4) (Fig. 1). For loads $N(t) > m^2N_e$, the sign of α_m^2 is positive.

The Cauchy problem of (4) for the inhomogeneous differential equation describes the amplitude of the motion as a function of time.

An equation analogous to (4) has been derived previously [4], which related to a load applied to a rod that increased in proportion to time. An asymptotic analysis has also been given [3] for an elastic rod subject to an aperiodic load, which varied slowly, in which case the equation for the amplitudes did not have turning points. Here we examine the motion of a rod for the general case of monotonically increasing loading, where there are turning points in (4).

The asymptote for the eigenvalues has been derived [5] for homogeneous second-order differential equations of the type of (4) containing $\alpha_m^2(t)$ and having first-order zeros for λ large; the essence of the method used to define the asymptotic representation is that the solution to the particular equation is expressed in terms of the solutions to a standard equation that represents precisely the behavior of the coefficients, namely, zeros of the same order. The standard-equation method is almost the same as the WBC method.

The standard-equation method is applied by means of a time shift in the loading function; we introduce the new variable $\tau = t - t_m$, whereupon all turning points shift to the one point $\tau = 0$; the homogeneous equation for (4) is then

$$q''_m(\tau) - \lambda^2\alpha_m^2(\tau)\,q_m(\tau) = 0. \tag{5}$$

As our standard equation corresponding to (5) we take the Airy equation

$$V''_m - (s - t_m)\,V_m = 0,\ V_m = V_m(s - t_m). \tag{6}$$

The Airy functions V_{m1}, and V_{m2} are linearly independent solutions to (6) and satisfy the initial conditions

$$V_{m1}(0) = 1,\ V'_{m1}(0) = 0,\ V_{m2}(0) = \Gamma(1/3)/3^{1/3}\Gamma(2/3),\ V'_{m2}(0) = 1$$

and the conditions at infinity

$$V_{m1} \to \infty,\ V_{m2} \to 0\ \text{for}\ \ s \to \infty.$$

The linearly independent solution to (5) is sought in terms of the Airy function V_{m1} as

$$q_{m1}(\tau) = A_m(\tau) V_{m1}[\varphi_m(\tau)]. \tag{7}$$

We substitute (7) into (5) to get

$$A_m(\tau)\left[V''_{m1}\varphi'^2_m - \lambda^2\alpha^2_m V_{m1}\right] + V'_{m1}\left[2A'_m\varphi'_m + A_m\varphi''_m\right] + V_{m1}A''_m = 0. \tag{8}$$

Equation (6) allows us to represent the expression in brackets for $A_m(\tau)$ as

$$V''_{m1}\varphi'^2_m - \lambda^2\alpha^2_m V_{m1} = V_{m1}\left(\varphi_m\varphi'^2_m - \lambda^2\alpha^2_m\right).$$

Therefore, we put

$$\varphi_m\varphi'^2_m = \lambda^2\alpha^2_m, \tag{9}$$

and eliminate the terms containing λ^2 in (8); we solve (9) to get

$$\varphi_m = \lambda^{2/3}\omega_m,\ \omega_m^{1/2}\omega'_m = \alpha_m,\ \omega_m = \left(\frac{3}{2}\int\limits_0^\tau \alpha_m d\tau\right)^{2/3}.$$

We equate the expression for V'_{m1} to zero to eliminate the terms of order $\lambda^{2/3}$ in (8); this gives the following equation for $A_m(\tau)$:

$$2A'_m\varphi'_m + A_m\varphi''_m = 0,$$

which itself gives

$$\frac{A'_m}{A_m} = -\frac{\varphi''_m}{2\varphi'_m} = -\frac{\omega''_m}{2\omega'_m},\ A_m(\tau) = \frac{c_{m1}}{(\omega'_m)^{1/2}}.$$

Another linearly independent solution to (5) is sought in terms of the Airy function V_{m2} as

$$q_{m2} = B_m(\tau)V_{m2}[\varphi_m(\tau)]\ (B_m(\tau) = c_{m2}(\omega'_m)^{-1/2}).$$

The expression in parentheses for $B_m(\tau)$ is derived by repeating the above arguments; $\varphi_m(\tau)$ as before satisfies (9).

As $A''_m \neq 0$ and $B''_m \neq 0$, unbalanced terms of order λ appear to power zero in (8), while terms of order $\lambda^{2/3}$ and λ^2 are eliminated. The general asymptotic solution to (5) is then

$$q_m(\tau) = c_{m1}(\omega'_m)^{-1/2}V_{m1}\left[\lambda^{2/3}\omega_m\right] + c_{m2}(\omega'_m)^{-1/2}V_{m2}\left[\lambda^{2/3}\omega_m\right] + O(1). \tag{10}$$

The known general solution to (10) for the homogeneous differential equation of (5) allowsus to find by variation the solution to the inhomogeneous problem of (4), which takes the following form in terms of the old variables:

$$q_m(t) = \frac{V_{m2}\left[\lambda^{2/3}\omega_m\right]}{\lambda^{2/3}(\omega'_m)^{1/2}}\int\limits_0^{t_m}\frac{f_m V_{m1}\left[\lambda^{2/3}\omega_m\right]dt}{(\omega'_m)^{1/2}} + \frac{V_{m2}\left[\lambda^{2/3}\omega_m\right]}{\lambda^{2/3}(\omega'_m)^{1/2}}\int\limits_{t_m}^{t}\frac{f_m V_{m1}\left[\lambda^{2/3}\omega_m\right]dt}{(\omega'_m)^{1/2}} -$$

$$-\frac{V_{m1}\left[\lambda^{2/3}\omega_m\right]}{\lambda^{2/3}(\omega'_m)^{1/2}}\int\limits_0^{t_m}\frac{f_m V_{m2}\left[\lambda^{2/3}\omega_m\right]dt}{(\omega'_m)^{1/2}} - \frac{V_{m1}\left[\lambda^{2/3}\omega_m\right]}{\lambda^{2/3}(\omega'_m)^{1/2}}\int\limits_{t_m}^{t}\frac{f_m V_{m2}\left[\lambda^{2/3}\omega_m\right]dt}{(\omega'_m)^{1/2}} + O(1). \tag{11}$$

Here the quantity $\lambda^{2/3}$ in the denominator is the value of the Wronskian for the functions $V_{m1}(\omega'_m)^{-1/2}$, and $V_{m2}(\omega'_m)^{-1/2}$ [5]:

$$\lambda^{2/3} = \frac{V_{m1}}{(\omega'_m)^{1/2}}\left[\frac{V_{m2}}{(\omega'_m)^{1/2}}\right]' - \frac{V_{m2}}{(\omega'_m)^{1/2}}\left[\frac{V_{m1}}{(\omega'_m)^{1/2}}\right]'.$$

Formula (11) describes the variation in the amplitudes; the motion is oscillatory as the load increases from zero up to $N_1 = N_e$ ($t < t_1$), but after passage through the turning point $t = t_1$ the amplitude for the first motion increases exponentially. The rod moves in the first form and vibrates in the second form with amplitude q_2 until $N(t)$ attains $N_2 = 4N_e$ ($t < t_2$); after the second point $t = t_2$, the amplitude of the second form of motion also increases exponentially. New forms of exponential motion appear when the load exceeds the corresponding static critical values. The more rapidly the load is applied, the more numerous the number of exponential forms of motion, and the more numerous the terms in (3) that have to be retained.

Therefore, we represent the main part of the solution for the motion when the load varies substantially as

$$w = \sum_{m=1}^{m_*} q_m(t) \sin \frac{m\pi x}{L}. \tag{12}$$

The functions $q_m(t)$ are defined by (11); the number of terms m_* is equal to the number of the critical Euler load attained by N(t), namely, $m_* = E \max \sqrt{N(t)/N_e}$, where E represents the integer part.

We now estimate the critical buckling time. It has been suggested [6] that the stability loss in an elastic shell subject to a varying load may be evaluated from the amplification factor for the initial irregularities. This factor is to be calculated for all forms of motion. The maximum value for some form defines the onset of stability loss.

We now consider the critical parameters: loading rate, buckling time, and amplification factor for an elastic rod subject to a slowly varying aperiodic load [3]. A difference from the method of [6] was that the form of stability loss chosen in [3] was a special one such that the coefficient was maximal in the exponent [1]. This represents a deliberate overestimate of the rate of increase of the deflection, which itself provides a lower bound to the critical time and critical loading rate. Practical calculations by the method of [3] are simpler, since there is no need to calculate the entire amplification curve. In the particular case of a constant heavy load, it becomes particularly simple to calculate the critical parameters [2].

We use the method of [3] with (11) and (12) to estimate the critical buckling time; a point about (12) is that each term consists of two factors, in which the second factor has a modulus not exceeding one. From (12) we have

$$|w| \leqslant \sum_{m=1}^{m_*} |q_m|. \tag{13}$$

The first term in (11) tends to zero for $t > t_m$ sufficiently large, since it contains an exponential with a large negative exponent before the definite integral.

We compare the second and fourth terms by replacing the Airy functions by the first terms in the asymptotic representations; the main term comes from the exponential if t is large (we omit factors less than 1). We have

$$J_1 = \exp\left(-\frac{2}{3}\lambda\omega_m^{3/2}\right)\int_{t_m}^{t} \exp\left(\frac{2}{3}\lambda\omega_m^{3/2}\right) dt,$$

$$J_2 = \exp\left(\frac{2}{3}\lambda\omega_m^{3/2}\right)\int_{t_m}^{t} \exp\left(-\frac{2}{3}\lambda\omega_m^{3/2}\right) dt.$$

We derive the integrals from the trapezium formula:

$$J_1 = [1 + \exp\{(-2/3)\lambda[\omega_m^{3/2}(t) - \omega_m^{3/2}(t_m)]\}](t - t_m)/2,$$
$$J_2 = [1 + \exp\{(2/3)\lambda[\omega_m^{3/2}(t) - \omega_m^{3/2}(t_m)]\}](t - t_m)/2.$$

These expressions show that the second term in (11) remains bounded as the time increases, whereas the force increases; as the sign of the second term differs from the signs of the third and fourth terms, we delete the second term and strengthen inequality (13):

$$|w| \leqslant \sum_{m=1}^{m_*} |q_m| \leqslant \sum_{m=1}^{m_*} \left| c_m V_{m1} \int_0^t V_{m2} dt \right|. \tag{14}$$

Here c_m is a constant determined by the conditions of the problem. We assume that the initial perturbations corresponding to various forms of motion are of the same order of smallness; we replace the Airy function in (14) by the first term in the asymptotic expansion and use the properties of the definite integral to get the normal deflection at time t_* as

$$|w(t_*)| \leqslant \sum_{m=1}^{m_*} \left| c_m \exp 1.5\,\lambda \int_{t_m}^{t_*} \alpha_m dt \max |V_{m2}|\, t_* \right| \leqslant m_* C t_* \sum_{m=1}^{m_*} \exp \alpha_* (t_* - t_m) \leqslant m_*^2 C t_* \exp (m_* \alpha_* t_*), \qquad (15)$$

$$C = 2,\ 1 \max|c_m|,\ \alpha_* = 0.75\lambda \max \alpha_m.$$

The factor $m_* \exp(m_* \alpha_* t_*)$ in (15) appears as a result of the estimate made for V_{m_1}: term m_* in the series, $\exp(m_* \alpha_* t_*)$, is summed m_* times. The $m_* C t_*$ factor represents an estimate of the integral with the coefficient from (14).

The critical time t_* is defined by

$$\max|w(t_*)| = w_0, \qquad (16)$$

if the maximum deflection is specified.

We substitute the estimate of (15) into (16) to get t_* as

$$w_0 = m_*^2 C t_* \exp(m_* \alpha_* t_*). \qquad (17)$$

This estimate for the critical buckling time is a complicated function, which differs from the result of [2, 3], since here we envisage a substantially varying load, and consequently the equation for the amplitudes has a turning point.

The right side of (17) is a rapidly increasing function; the coefficient in the exponent and the preexponential factor contain the large quantity m_*, which characterizes the loading rate. Therefore, we have lower bounds to the critical time, as in [2, 3]. Further, the estimate of (17) becomes even lower as the loading rate increases, which improves the reliability of practical calculations.

We have derived (11) above for the amplitude as a function of time; Fig. 2 shows calculations on the amplitude for the following values of the parameters: L = 800 mm, diameter 10 mm, $\lambda = 193\ \mathrm{sec}^{-1}$, Ne = 151.2 kgf, loading law N(t) = 1.7 exp 500 t; the calculations have been based on the tables of [7].

The ordinate is a dimensionless parameter equal to the ratio of q_m to the initial deflection f_m, while the abscissa is time t. Curves 1-3 correspond to m of 1, 2, and 3. There is a marked increase in the deflection after the turning point, and the various forms of motion diverge at different rates. The larger numbers correspond to larger rates of increase in the amplitude. The amplitudes are of the same order, so all must be incorporated. Therefore, this system with distributed parameters may be represented as one with a large but finite number of degrees of freedom when the load increases monotonically.

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